STAT 2593

Lecture 016 - Probability Distribution Functions

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Probability Distribution Functions

Learning Objectives

1. Understand probability density functions for continuous variables.

2. Understand the uniform distribution and its properties.


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SPEED STACKS

The Problem with Continuous Random Variables

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  - What will happen to $P(\mathcal{S})$?
- Recall that we also had a **cumulative distribution function** for $P(X \leq x)$.
  - This is still well-defined for continuous random variables.

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  - By the **fundamental theorem of calculus** this means that $f(x) = \frac{d}{dx}F_X(x)$ is analogous to the pmf.
- We call $f(x)$ the **probability density function** (PDF).

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- We must have that $f(x) \geq 0$ for all x and that $\int_{-\infty}^{\infty} f(x)dx = 1$.

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- Note that $P(X = c) = 0$ for any c, even if $c \in [a, b]$.
- If we have $[c, d] \subset [a, b]$ then the probability that $X \in [c, d]$ is

$$\int_c^d \frac{1}{b-a} dx = \frac{d-c}{b-a}.$$

Summary

- Continuous random variables do not have probability mass functions, they have probability density functions.
- A PDF can be integrated to determine the probability of an event.
- Singletons have probability 0.
- The uniform distribution is a continuous distribution which gives equal density to all values on an interval.